Complex Numbers

$$\sqrt{-1} = i$$

	4 or 6 Marks	
1)	Z1 = 4 – 5i Z2 = 3 + 7i Find |Z1 +Z2| **Solution:** $z_1 + z_2 = 4 - 5i + 3 + 7i$ $\therefore z_1 + z_2 = 7 + 2i$ $\therefore \lvert z_1 + z_2 \rvert = \sqrt{(7)^2 + (2)^2}$ $\therefore \lvert z_1 + z_2 \rvert = \sqrt{49 + 4}$ $\therefore \lvert z_1 + z_2 \rvert = \sqrt{53}$	
2)	Express in polar form Z = 1 – i **Solution:** $z = 1 - i$ $\therefore x = 1, y = -1$ $\therefore r = \lvert z \rvert = \sqrt{x^2 + y^2} = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}$ $\theta = 2\pi - \tan^{-1}\left(\left\lvert \frac{y}{x} \right\rvert\right) = 2\pi - \tan^{-1}\left(\left\lvert \frac{-1}{1} \right\rvert\right) = 2\pi - \tan^{-1}(1)$ $\theta = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$ $\therefore$ Polar form is $z = r(\cos\theta + i\sin\theta)$ $\therefore\ 1 - i = \sqrt{2}\left(\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)\right)$	

3)	Express in a+ib form $Z=\frac{1-i}{1+i}$ **Solution :** $z=\frac{1-i}{1+i}$ $\therefore z=\frac{1-i}{1+i}\times\frac{1-i}{1-i}$ $\therefore z=\frac{1-2i+i^2}{1^2-i^2}$ $\therefore z=\frac{1-2i-1}{1+1}$ $\therefore z=\frac{-2i}{2}$ $\therefore z=-i=0-i$	
4)	Express in a+ib form $Z=\frac{1+i}{2-i}$ $Z=\frac{1+i}{2-i}$ $=\frac{1+i}{2-i}\times\frac{2+i}{2+i}$ $=\frac{2+i+2i+i^2}{2^2-i^2}$ $=\frac{2+3i-1}{4+1}$ $=\frac{1+3i}{5}$ $=\frac{1}{5}+\frac{3i}{5}$	

5)	Express in polar form $Z = \frac{1+i}{2-i}$	
6)	Express in polar form $Z = 1 + i\sqrt{3}$ **Solution:** $Z = 1 + i\sqrt{3}$ Comparing with $Z = x + iy$ $\therefore x = 1, y = \sqrt{3}$ $r = \sqrt{x^2 + y^2}$ $r = \sqrt{(1)^2 + (\sqrt{3})^2} = 2$ $\theta = \tan^{-1}\left\|\frac{y}{x}\right\|$ $\theta = \tan^{-1}\left\|\frac{\sqrt{3}}{1}\right\|$ $\theta = \tan^{-1}\sqrt{3}$ $\theta = 60^0$ or $\frac{\pi}{3}$ In polar form, $Z = r(\cos\theta + i\sin\theta)$ $Z = 2(\cos 60^0 + i\sin 60^0)$ or $Z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$	

7)	If $Z_1 = -3+4i$, $Z_2 = 5-3i$ Express $\frac{Z1}{Z2}$ in x + *i*y format **Solution:** $\frac{z_1}{z_2}=\frac{-3+4i}{5-3i}$ $\therefore \frac{z_1}{z_2}=\frac{-3+4i}{5-3i}\times\frac{5+3i}{5+3i}$ $\therefore \frac{z_1}{z_2}=\frac{-15-9i+20i+12i^2}{25-9i^2}$ $\therefore \frac{z_1}{z_2}=\frac{-15-9i+20i+12(-1)}{25-9(-1)}$ $\therefore \frac{z_1}{z_2}=\frac{-27+11i}{34}$ $\therefore \frac{z_1}{z_2}=\frac{-27}{34}+\frac{11}{34}i$	
8)	Express in x + *i*y form $\frac{2-i\sqrt{3}}{1+i}$ **Solution:** $\frac{2-\sqrt{3}i}{1+i}$ $=\frac{2-\sqrt{3}i}{1+i}\times\frac{1-i}{1-i}$ $=\frac{2-2i-\sqrt{3}i+\sqrt{3}i^2}{1-i^2}$ $=\frac{2-(2+\sqrt{3})i+\sqrt{3}(-1)}{1-i^2}$ $=\frac{2-(2+\sqrt{3})i-\sqrt{3}}{1+1}$ $=\frac{(2-\sqrt{3})-(2+\sqrt{3})i}{2}$ $=\frac{(2-\sqrt{3})}{2}-\frac{(2+\sqrt{3})i}{2}$	

9)	If $w_1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ $w_2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$ Then show that $w_1^2 = w_2$ **Solution:** $\omega_1^2 = \left(\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right)^2$ $= \frac{1}{4} - 2\left(\frac{1}{2}\right)\left(i\frac{\sqrt{3}}{2}\right) + i^2\left(\frac{\sqrt{3}}{2}\right)^2$ $= \frac{1}{4} - i\frac{\sqrt{3}}{2} - \frac{3}{4}$ $= \frac{-1}{2} - i\frac{\sqrt{3}}{2}$ $= \omega_2$ $\therefore \omega_1^2 = \omega_2$	

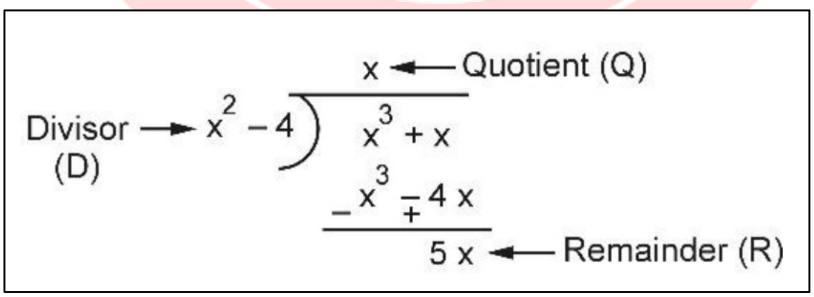
Laplace formulas

Sr No.	f(t)	L{f(t)}
1.	K = Constant	$\frac{K}{s}, \quad s > 0$
2.	e^t	$\frac{1}{s-a}, \quad s > a$
3.	t^n, n = 0,1,2,3…. Integer	$\frac{n!}{s^{n+1}}, \quad s > 0$
4.	sin (kt)	$\frac{k}{s^2+k^2}, \quad s > 0$
5.	cos (kt)	$\frac{s}{s^2+k^2}, \quad s > 0$
6.	sinh (kt)	$\frac{k}{s^2-k^2}, \quad s^2 > k^2$
7.	cosh (kt)	$\frac{s}{s^2-k^2}, \quad s^2 > k^2$
	First Translation of Laplace	
1.	L (e^{at}. t^n)	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
2.	L (e^{at}. Sin kt)	$\frac{k}{(s-a)^2+k^2}, \quad s > a$
3.	L (e^{at}. Cos kt)	$\frac{s-a}{(s-a)^2+k^2}, \quad s > a$
4.	L (e^{at}. Sinh kt)	$\frac{k}{(s-a)^2-k^2}, \quad s > a$
5.	L (e^{at}. Cosh kt)	$\frac{s-a}{(s-a)^2-k^2}, \quad s > a$

Laplace Inverse formulas

1.	$L^{-1}\left\{\frac{1}{k}\right\} = k$
2.	$L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$
3.	$L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^n}{(n-1)!}$
4.	$L^{-1}\left\{\frac{1}{(s+a)^n}\right\} = \frac{e^{-at}.t^{n-1}}{(n-1)!}$
5.	$L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}.\sin(at)$
6.	$L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos(at)$
7.	$L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh(at)$
8.	$L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh(at)$
9.	$L^{-1}\left\{\frac{1}{(s+a)^2+k^2}\right\} = \frac{1}{k}.e^{-at}.\sin(kt)$
10.	$L^{-1}\left\{\frac{s+a}{(s+a)^2-k^2}\right\} = e^{-at}.\cos(kt)$
11.	$L^{-1}\left\{\frac{1}{(s+a)^2-k^2}\right\} = \frac{1}{k}.e^{-at}.\sinh(kt)$
12.	$L^{-1}\left\{\frac{s+a}{(s+a)^2-k^2}\right\} = e^{-at}.\cosh(kt)$

Methods of Partial fraction

1.	**Case 1:** $\dfrac{ax+b}{(x-\alpha)(x-\beta)(x-\gamma)} = \dfrac{A}{(x-\alpha)} + \dfrac{B}{(x-\beta)} + \dfrac{C}{(x-\gamma)}$
2.	**Case 2:** $\dfrac{ax+b}{(x-\alpha)(x-\beta)^2} = \dfrac{A}{(x-\alpha)} + \dfrac{B}{(x-\beta)} + \dfrac{C}{(x-\beta)^2}$
3.	**Case 3:** $\dfrac{ax+b}{(x-\alpha)(x^2+\beta)} = \dfrac{A}{(x-\alpha)} + \dfrac{Bx+C}{(x^2+\beta)}$
4.	**Case 4:** $\dfrac{x^m+a}{x^n+b} = Quotient + \dfrac{Reminder}{Divisor}$ 

Important Questions of Laplace

	3 Marks	
a)	$L\{\sin 3t + \cos 2t\}$	
b)	$L\{e^{-4t}t^2\}$	
c)	$L\{e^{-3t}\sin(2t)\}$	
d)	$L\{e^3t(t^2 + t)\}$	
	4 or 6 Marks	
a)	$L^{-1}\left\{\frac{2s+3}{(s+2)(s+6)}\right\}$	
b)	$L^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\}$	
c)	$L^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}$	
d)	$L^{-1}\left\{\frac{s+2}{s^2-2s+5}\right\}$	
e)	$L^{-1}\left\{\frac{s-3}{s^2+2s+5}\right\}$	

	3 Marks	
a)	$L\{\sin 3t + \cos 2t\}$ **Solution:** $L\{\sin 3t + \cos 2t\}$ $= \frac{3}{s^2+3^2} + \frac{s}{s^2+2^2}$ $= \frac{3}{s^2+9} + \frac{s}{s^2+4}$	
b)	$L\{e^{-4t}t^2\}$ **Solution:** $L\{e^{-3t}\sin 2t\}$ $L\{\sin 2t\} = \frac{2}{s^2+2^2} = \frac{2}{s^2+4}$ $\therefore L\{e^{-3t}\sin 2t\} = \frac{2}{(s+3)^2+4}$ $\therefore L\{e^{-3t}\sin 2t\} = \frac{2}{s^2+6s+9+4}$ $\therefore L\{e^{-3t}\sin 2t\} = \frac{2}{s^2+6s+13}$	

c)	$L\{e^{-3t}\sin(2t)\}$ **Solution:** $\mathrm{L}\left\{e^3 t\left(t^2+t\right)\right\}$ $=\mathrm{L}\left\{e^3\left(t^3+t^2\right)\right\}$ $=e^3\mathrm{L}\left\{t^3+t^2\right\}$ $=e^3\left\{L\left(t^3\right)+L\left(t^2\right)\right\}$ $=e^3\left(\frac{3!}{s^{3+1}}+\frac{2!}{s^{2+1}}\right)$ $=e^3\left(\frac{6}{s^4}+\frac{2}{s^3}\right)$	
d)	$L\{e^3 t(t^2+t)\}$ **Solution:** $\mathrm{L}\left\{e^{-4t}t^2\right\}$ $L\left\{t^2\right\}=\frac{2!}{s^3}=\frac{2}{s^3}$ $\therefore \mathrm{L}\left\{e^{-4t}t^2\right\}=\frac{2}{(s+4)^3}$	

	4 or 6 Marks	
a)	$L^{-1}\left\{\frac{2s+3}{(s+2)(s+6)}\right\}$ **Solution:** $L^{-1}\left\{\frac{2s+3}{(s+2)(s+6)}\right\}$ Let $\frac{2s+3}{(s+2)(s+6)}=\frac{A}{s+2}+\frac{B}{s+6}$ $\therefore 2s+3=(s+6)A+(s+2)B$ Put $s=-2$ $\therefore -1=4A$ $\therefore A=-\frac{1}{4}$ Put $s=-6$ $-9=-4B$ $\therefore B=\frac{9}{4}$ $\therefore \frac{2s+3}{(s+2)(s+6)}=\frac{\frac{-1}{4}}{s+2}+\frac{\frac{9}{4}}{s+6}$ $\therefore L^{-1}\left\{\frac{2s+3}{(s+2)(s+6)}\right\}=-\frac{1}{4}L^{-1}\left\{\frac{1}{s+2}\right\}+\frac{9}{4}L^{-1}\left\{\frac{1}{s+6}\right\}$ $=-\frac{1}{4}e^{-2t}+\frac{9}{4}e^{-6t}$	

b) $L^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\}$

Solution:

Let

$$\frac{3s+1}{(s-1)(s^2+1)}=\frac{A}{s-1}+\frac{Bs+C}{s^2+1}$$

$$3s+1=(s^2+1)A+(s-1)(Bs+C)$$

Put $s=1$

$\therefore A=2$

Put $s=0$

$1=A+(-1)C$

$\therefore 1=2-C$

$\therefore C=1$

Put $s=-1$

$-2=2A+(-2)(-B+C)$

$\therefore -2=2(2)+2B-2(1)$

$\therefore -2=2+2B$

$\therefore B=-2$

$$\therefore \frac{3s+1}{(s-1)(s^2+1)}=\frac{2}{s-1}+\frac{-2s+1}{s^2+1}$$

$$\therefore L^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\}=2L^{-1}\left\{\frac{1}{s-1}\right\}-2L^{-1}\left\{\frac{s}{s^2+1}\right\}+L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$=2e^t-2\cos t+\sin t$$

c) $L^{-1}\left\{\dfrac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}$

Solution:

$$\frac{2s^2-4}{(s+1)(s-2)(s-3)}=\frac{A}{s+1}+\frac{B}{s-2}+\frac{C}{s-3}$$

$$2s^2-4=(s-2)(s-3)A+(s+1)(s-3)B+(s+1)(s-2)C$$

Put $s=-1$

$$\therefore 2(-1)^2-4=(-1-2)(-1-3)A$$

$$\therefore A=-\frac{1}{6}$$

Put $s=2$

$$2(2)^2-4=(2+1)(2-3)B$$

$$\therefore B=\frac{-4}{3}$$

Put $s=3$

$$2(3)^2-4=(3+1)(3-2)C$$

$$\therefore c=\frac{7}{2}$$

$$\therefore \frac{2s^2-4}{(s+1)(s-2)(s-3)}=\frac{-\frac{1}{6}}{s+1}+\frac{\frac{-4}{3}}{s-2}+\frac{\frac{7}{2}}{s-3}$$

$$\therefore L^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}=-\frac{1}{6}L^{-1}\left\{\frac{1}{s+1}\right\}-\frac{4}{3}L^{-1}\left\{\frac{1}{s-2}\right\}+\frac{7}{2}L^{-1}\left\{\frac{1}{s-3}\right\}$$

$$=-\frac{1}{6}e^{-t}-\frac{4}{3}e^{2t}+\frac{7}{2}e^{3t}$$

d) $L^{-1}\left\{\frac{s+2}{s^2-2s+5}\right\}$

Solution:

(e) $L^{-1}\left\{\frac{s+2}{s^2-2s+5}\right\}$

$= L^{-1}\left\{\frac{s+2}{s^2-2s+1+4}\right\}$

$= L^{-1}\left\{\frac{(s-1)+3}{(s-1)^2+(2)^2}\right\}$

$= L^{-1}\left\{\frac{s-1}{(s-1)^2+(2)^2}\right\} + 3\,L^{-1}\left\{\frac{1}{(s-1)^2+(2)^2}\right\}$

$= e^t \cdot \cos 2t + 3 \cdot \frac{1}{2} \cdot e^t \sin 2t$

$= e^t \cdot \left(\cos 2t + \frac{3}{2}\sin 2t\right)$

e) $L^{-1}\left\{\frac{s-3}{s^2+2s+5}\right\}$

Solution:

(b) $L^{-1}\left\{\frac{s-3}{s^2+2s+5}\right\}$

$= L^{-1}\left\{\frac{s-3}{s^2+2s+1+4}\right\}$

$= L^{-1}\left\{\frac{s-3}{(s+1)^2+4}\right\}$

$= L^{-1}\left\{\frac{s+1-4}{(s+1)^2+4}\right\}$

$= L^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\} - 4 \cdot L^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\}$

$= e^{-t}\cos 2t - 4 \cdot \frac{1}{2} \cdot e^{-t}\sin 2t$

$= e^{-t}(\cos 2t - 2\sin 2t)$